Lecture 18:

Recall: Power method:

$$\vec{x}^{(k+1)} = \frac{A\vec{x}^{(k)}}{\|A\vec{x}^{(k)}\|_\infty}$$

$$\Downarrow$$

$$\vec{x}^{(k)} = \frac{A^k\vec{x}^{(k)}}{\|A\vec{x}^{(k)}\|_\infty}$$

Note: $\|A\vec{x}^{(k)}\|_\infty \to |\lambda_1|$ if

$$\vec{x}^{(0)} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n \quad (c_1 \neq 0)$$

where $\{\vec{v}_1, \dots, \vec{v}_n\}$ = basis of eigenvectors.

How about if A is NOT diagonalizable?

(Jordan Canonical Form)

Let $A = VJV^{-1}$, J = Jordan Canonical Form

Since the dominant eigenvalue, λ_1, has multiplicity 1, the first Jordan block of A must be a 1×1 matrix.

$$\therefore \quad J = \begin{pmatrix} \lambda_1 & & & \\ & [J(\lambda_{i_2})] & & \\ & & \ddots & \\ & & & [J(\lambda_{i_k})] \end{pmatrix}$$

Recall:

$$J(\lambda_j) = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

Let $V = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$ and let $\vec{x}^{(0)} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$

Assume $c_1 \neq 0$, then:

$$\vec{x}^{(k)} = \frac{A^k \vec{x}^{(0)}}{\| A^k \vec{x}^{(0)} \|_\infty} = \frac{(VJV^{-1})^k \vec{x}^{(0)}}{\| (VJV^{-1})^k \vec{x}^{(0)} \|_\infty}$$

$$= \frac{(VJ^kV^{-1})(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n)}{\|(VJ^kV^{-1})(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n)\|_\infty}$$

(note: $\vec{v}_1$ = eigenvector of λ_1)

$$= \frac{VJ^k(c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n)}{\|VJ^k(c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n)\|_\infty}$$

($\because$ $V^{-1}\vec{v}_j = \vec{e}_j = \begin{array}{r} \\ \\ \\ j^{th} \rightarrow \\ \\ \\ \\ \end{array}\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$)

$$= \frac{c_1\lambda_1^k\left(\vec{v}_1 + \frac{1}{c_1}V\left(\frac{1}{\lambda_1}J\right)^k(c_2\vec{e}_2 + \dots + c_n\vec{e}_n)\right)}{\left\|c_1\lambda_1^k\left(\vec{v}_1 + \frac{1}{c_1}V\left(\frac{1}{\lambda_1}J\right)^k(c_2\vec{e}_2 + \dots + c_n\vec{e}_n)\right)\right\|_\infty}$$

(Note: $J^k\vec{e}_1 = \lambda_1^k\vec{e}_1$
$\Rightarrow VJ^k\vec{e}_1 = \lambda_1^k\vec{v}_1$)

Now, $\left(\frac{1}{\lambda_1} J\right)^k = \begin{pmatrix} 1 & & & \\ & \left[\frac{1}{\lambda_1} J(\lambda_{i_2})\right]^k & & \\ & & \ddots & \\ & & & \left[\frac{1}{\lambda_1} J(\lambda_{i_k})\right]^k \end{pmatrix} \to \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$ as $k \to \infty$

$\left|\frac{\lambda_{i_2}}{\lambda_1}\right| < 1$ $\quad$ $\left|\frac{\lambda_{i_k}}{\lambda_1}\right| < 1$

$\therefore \quad \vec{x}^{(k)} \approx \frac{c_1 \lambda_1^k \vec{v}_1}{\|c_1 \lambda_1^k \vec{v}_1\|_\infty}$ when k is large.

$\therefore \quad \|A \vec{x}^{(k)}\|_\infty \to |\lambda_1|$ as $k \to \infty$ as before!

Generalization of Power method

Consider an invertible matrix A. Suppose A has eigenvalues:

$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n| \ (>0)$

Consider A^{-1} (exist as all eigenvalues are non-zero). Then A^{-1} has eigenvalues:

$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ with $\left|\frac{1}{\lambda_n}\right| > \left|\frac{1}{\lambda_{n-1}}\right| > \ldots > \left|\frac{1}{\lambda_1}\right|$

<u>Extension</u>: Apply Power's method on A^{-1} to obtain $\left|\frac{1}{\lambda_n}\right|$.

$\therefore$ the minimal eigenvalue can be determined! (Inverse Power method)

<u>Remark</u>: Computing A^{-1} is difficult! We solve: $A\vec{y} = \vec{x}^{(n)}$ in each iteration to determine $A^{-1}\vec{x}^{(n)}$.

Finding A^{-1} is equivalent to solving:

$A\vec{y} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ A\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \ A\vec{y} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$

$$A \underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}}_{A^{-1}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & & \vdots \\ & \vdots & & \\ 0 & 0 & & 1 \end{pmatrix}$$

Algorithm: (Inverse Power method)

Step 1: Pick $\vec{x}^{(0)}$ with $\|\vec{x}^{(0)}\|_\infty = 1$

Step 2: For $k = 1, 2, \ldots,$ solve $A\vec{w} = \vec{x}^{(k-1)}$.

Let $\vec{x}^{(k)} = \dfrac{\vec{w}}{\|\vec{w}\|_\infty}$.

Let $\rho_k = \|A\vec{x}^{(k)}\|_\infty$.

Remark: Again, $\rho_k \to |\lambda_n|$ as $k \to \infty$. ($\vec{x}^{(k)} \approx$ eigenvector of eigenvalue λ_n)

Lecture 18:

Inverse power method with shift

Goal: Take $\mu \in \mathbb{R}$. Find the eigenvalue of A closest to μ.

Observation: Consider $B = A - \mu I$. Then B has eigenvalues:

$$\{\lambda_1 - \mu, \lambda_2 - \mu, \ldots, \lambda_n - \mu\} \leftarrow$$

Inverse Power method find eigenvalues such that $|\lambda_j - \mu|$ is the smallest.

$\therefore$ λ_j closest to μ can be found.

Algorithm: (Inverse power method with shift)

Step 1: Take $\mu \in \mathbb{R}$. Pick $\vec{x}^{(0)}$ such that $\|\vec{x}^{(0)}\|_\infty = 1$.

Step 2: For $k = 1, 2, \ldots$

Solve: $(A - \mu I)\vec{w} = \vec{x}^{(k-1)}$ for $\vec{w}$.

Let: $\vec{x}^{(k)} = \dfrac{\vec{w}}{\|\vec{w}\|_\infty}$.

Let $\rho_k = \|A\vec{x}^{(k)}\|_\infty$ ($\rho_k \to |\lambda_j|$ as $k \to \infty$)